

RESTRICTED SUMS OF FOUR INTEGRAL SQUARES

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ABSTRACT. We give a simple quaternionic proof of a recent result of Goldmakher and Pollack on restricted sums of four integral squares.

In [1] the authors prove the following result:

Theorem 1 (Goldmakher, Pollack). *Let $n \equiv T \pmod{2}$ be integers. Then n has a representation $n = \sum_{\nu=0}^3 a_\nu^2$ as a sum of four integer squares with $\sum_{\nu=0}^3 a_\nu = T$ if and only if $4n - T^2$ is a sum of three integral squares.*

We give here a different proof using a very simple computation in the ring of integral quaternions.

Proof. Let $1, i, j, k$ denote the usual basis of the Hamilton quaternions \mathbb{H} , for $\alpha = a_0 + a_1i + a_2j + a_3k \in \mathbb{H}$ write $\operatorname{Re}(\alpha) := a_0$, $N(\alpha) := \sum_{\nu=0}^3 a_\nu^2$, $\varphi(\alpha) := \sum_{\nu=0}^3 a_\nu$. We put $R = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ and define $f : R \rightarrow R$ by $f(\alpha) = \alpha(1 - i - j - k)$. We have $\operatorname{Re}(f(\alpha)) = \varphi(\alpha)$, $N(f(\alpha)) = 4N(\alpha)$, and since $N(\alpha) \equiv \varphi(\alpha) \pmod{2}$ for $\alpha \in R$, we have $f(R) \subseteq X := \{\beta \in R \mid N(\beta) \equiv 4\operatorname{Re}(\beta) \pmod{8}\}$.

Since $(1 - i - j - k)^{-1} = (1 + i + j + k)/4$ we see that $f(R) = \{\beta \in R \mid \beta(1 + i + j + k)/4 \in R\} = \{\beta = b_0 + b_1i + b_2j + b_3k \in R \mid \sum_{\nu \neq \mu} b_\nu \equiv b_\mu \pmod{4} \text{ for } 0 \leq \mu \leq 3\}$.

For $\beta' = b'_0 + b'_1i + b'_2j + b'_3k \in X$ one easily sees that the b'_ν are all congruent modulo 2 and if they are even, the number of $b'_\nu \equiv 2 \pmod{4}$ is 0, 2 or 4. In the latter case, the congruence condition $\sum_{\nu \neq \mu} b'_\nu \equiv b'_\mu \pmod{4}$ for $0 \leq \mu \leq 3$ is satisfied, if the b'_ν are all odd, it is satisfied if either one or three of them are congruent to -1 modulo 4. In both cases we find, changing a sign if necessary, a $\beta = b_0 + b_1i + b_2j + b_3k$ with $b_0 = b'_0$, $b_\nu \in \{b'_\nu, -b'_\nu\}$ for all ν satisfying the congruence condition above, hence $\beta \in f(R)$.

Taken together we obtain for $T \equiv n \pmod{2}$: There exists $\alpha = a_0 + a_1i + a_2j + a_3k \in R$ with $N(\alpha) = \sum_{\nu=0}^3 a_\nu^2 = n$ and $\varphi(\alpha) = \sum_{\nu=0}^3 a_\nu = T$ if and only if there exists $\beta = b_0 + b_1i + b_2j + b_3k \in X$ with $b_0 = T$, which is again equivalent to the existence of a representation of $4n - T^2$ as a sum of three squares, and the theorem is proved. \square

REFERENCES

- [1] L. Goldmakher, P. Pollack: Refinement of Lagrange's four square theorem, matharxiv 1703.03092

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